Modern Statistics **Lecture 3 - 2/24/2025**

Random Variable & Vector

Lecturer:Xiangyu Chang *Scribe: Yuqi Meng, Yupeng Tong*

Edited by: Zhihong Liu

1 Recall

- **Ω: Sample space**, the set of possible outcomes of experiments.
- **A: Events**, the subset of Ω ($A \subset \Omega$).
- **Probability**: The concept "probability" can be understood as a function (or a mapping).

  Definition 1 *P: $A \subseteq \Omega \to [0,1]$ that satisfies 3 axioms:*

  - $P(A) \geq 0, \forall A \subseteq \Omega$.
  - $P(\Omega) = 1$.
  - $\{A_i\}_{i=1}^n, A_i \cap A_j = \emptyset, P(\bigcap\limits_{i=1}^{n} A_i) = \sum_{i=1}^n P(A_i)$.

- **Random Variable: A random variable** X is a measurable function from a probability space $(\Omega, \mathcal{F}, P)$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$:

$$X : \Omega \to \mathbb{R}$$

  - X: is a mapping, measurable function.
  - Ω: outcomes.
  - $\omega \in \Omega, X(\omega) \in \mathbb{R}$.

- **the Probability of Random Variable:**
  - $X : \Omega \to \mathbb{R}$**. An outcome in the sample space is defined as a real number.**
  - **Define a set** $A = [a, b] \subseteq \mathbb{R}$**, then** $P(X \in A) = P(\{\omega | \omega \in \Omega, X(\omega) \in A\})$**.**

- **CDF(Cumulative Distribution Function) : For a random variable** X**, its Cumulative Distribution Function is the function** $F_X : \mathbb{R} \to [0, 1]$ **defined by:**

$$F_X(x) = P(X \leq x) = P(\{\omega \in \Omega \mid X(\omega) \leq x\}).$$

2 PDF (probability distribution function)

Definition 2 *x is continuous, if there exists $f_X(x)$.*

- $\int_{\mathbb{R}} f_X(x)dx = 1$.
- $P(a < x < b) = \int_a^b f(x)dx$. (proof:Integral mean value theorem)

Definition 3 (Quantile) $F_X^{-1}(q) \triangleq \text{Inf}\{x|F_X(x) > q\}$

- ***Median***: $F_X^{-1}(\frac{1}{2})$
- ***First Quartile***: $F_X^{-1}(\frac{1}{4})$
- ***Third Quartile***: $F_X^{-1}(\frac{3}{4})$

Example 1

$$f_X(x)) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & others \end{cases} \quad F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases} \tag{1}$$

- *Median:* $F_X^{-1}(\frac{1}{2}) = \frac{1}{2}$.
- *First Quartile:* $F_X^{-1}(\frac{1}{4}) = \frac{1}{4}$.
- *Third Quartile:* $F_X^{-1}(\frac{3}{4}) = \frac{3}{4}$.

Definition 4 *X is discrete if X countably takes from $\{x_1, x_2, \cdots, x_n\}$*

$$PMF(Probability\ Mass\ Function) : f_X(x) \triangleq \text{P}(X = x)$$

Example 2 $f_X(k) = P(X = k) = C_k^n(\frac{1}{2})^n, 0 \leq k \leq n$

3 Important R.V.

1. Point Mass
$$P(X = a) = 1, F_X(x) = \begin{cases} 0, & x < a \\ 1, & x \geq a \end{cases}$$
2. Discrete Uniform Distribution
$$X \in \{1, \cdots, n\}, P(X = k) = \frac{1}{n}, k \in \{1, 2, \cdots, n\}$$

3. Bernoulli

$$X \in \{0,1\}, P(X=x) = p^x(1-p)^{1-x}$$

4. Binomial(n,p)

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

- Additivity: if $X_1 \sim Bin(n_1,p), X_2 \sim Bin(n_2,p)$, then $Z = X_1 + X_2 \sim Bin(n_1+n_2,p)$, where X_1 and X_2 are independent.
- The sum of Bernoulli trials: if $Z \sim Bin(n,p)$, $X_i \sim Ber(p)$, then $Z = \sum_{i=1}^{n} X_i$.

5. Poisson(λ):

$$f(x) = P(X=x) = e^{-\lambda}\frac{\lambda^x}{x!} \quad (x = 0,1,\cdots)$$

$$Verification: \sum_{k=0}^{\infty} P(X=k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

6. Geometric:

$$P(X=x) = p(1-p)^x \quad (x \geq 0)$$

7. Continuous Uniform Distribution:

$$X \sim U(a,b),\ f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & \text{otherwise} \end{cases}$$

8. Normal Distribution:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ X \sim N(\mu,\sigma^2) \quad (\mu\ is \text{ mean },\ \sigma^2 is \text{ variance })$$

- Standard Normal:

$$X \sim N(0,1) \Rightarrow P(X < x) = \varPhi(x)$$

- If $Z \sim N(\mu,\sigma^2)$, $Z = \mu + \sigma X$, then $X \sim N(0,1)$.

4 Transformation of R.V.

Suppose $Y = g(X)$,

1. $f_Y(y) = P(Y=y) = P(g(X)=y) = P(X = g^{-1}(y))$, when $g(X)$ is monotonic.
2. $F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$, when $g(X)$is monotonic, $P(g(X) \leq y) = P(X \leq g^{-1}(y)) = \int_{-\infty}^{g^{-1}(y)} f_X(x)dx$, $f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$.
3. If $X \sim N(\mu,\sigma^2)$ Let $Z = \frac{X-\mu}{\sigma}$, then $P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = \varPhi\left(\frac{x-\mu}{\sigma}\right)$ ($\varPhi(\cdot)$ is the cumulative - distribution function of the standard normal distribution).

4. If X_i are independent of each other and$X_i \sim N(\mu_i, \sigma_i^2)$, then $X = \sum_{i=1}^k X_i \sim N\left(\sum_{i=1}^k \mu_i, \sum_{i=1}^k \sigma_i^2\right)$.

Example 3 *if* $Z = \mu + \sigma X$, *then* $g^{-1}(z) = \frac{z-\mu}{\sigma}$, $\frac{dg^{-1}(z)}{dz} = \frac{1}{\sigma}$, *so* $f_Z(z) = f_X\left(\frac{z-\mu}{\sigma}\right)\left|\frac{1}{\sigma}\right|$. *If* $X \sim N(0,1)$,$f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, *then* $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(z-\mu)^2}{2\sigma^2}}$, $Z \sim N(\mu, \sigma^2)$.

5 Bivariable

Definition 5

$$PMF: Let\, Z = (X,Y)^T \in \mathbb{R}^2, f_Z(z) = P(X = x, Y = y)$$

Definition 6 (Marginal Distribution)

$$P(X) = \sum_y P(X = x, Y = y),\ P(X = x_i) = \sum_j p_{ij} = p_{i\cdot}$$

Definition 7 (PDF)

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

Property:

- $\int_{\mathbb{R}^2} f_{X,Y}(x,y)dxdy = 1$
- $\int_{(x,y)\in\mathbb{R}^2} f_{X,Y}(x,y)dxdy = P((X,Y) \in A)$

Definition 8 (Independence)

$$P(X \in A, Y \in B) = P(X \in A)\cdot P(Y \in B) \Leftrightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

Transformation If $y = g(x)$, then $f_Y(y) = f_X(g^{-1}(y)) \cdot |\det(\nabla g^{-1}(y))|$.

*Verification: Given X and $Y = F_X(X)$, try to prove $Y \sim U(0,1)$

Prove: $P(Y \le y) = P(F_X(X) \le y) = P(X \le F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$.