Modern Statistics **Lecture 1 - 2/17/2025**

Probability

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1 Sample Space and Events

Definition 1 (Sample Space) *The* ***sample space*** Ω *is the set of all possible outcomes of a random experiment. Elements* $\omega \in \Omega$ *are called* ***sample points****.*

Definition 2 (Event) *An* ***event*** A *is any subset of the sample space* ($A \subseteq \Omega$)*. We say event* A *occurs if the outcome* $\omega \in A$*.*

Example 1 (Coin Tossing) *Let's analyze different coin tossing scenarios:*

- ***Single toss****:* $\Omega = \{H, T\}$
- ***Two tosses****:* $\Omega = \{HH, HT, TH, TT\}$
- ***Conditional case****: When requiring the first toss to be heads:* $\Omega_{cond} = \{HH, HT\}$

Example 2 (Infinite Case) *For an infinite sequence of coin tosses:*

$$\Omega = \{(\omega_1, \omega_2, \omega_3, \ldots) : \omega_i \in \{H, T\}\}$$

2 Set Operations and Limits

2.1 Basic Operations

- **Intersection**: $A \cap B = \{\omega \in \Omega | \omega \in A \text{ and } \omega \in B\}$
- **Union**: $A \cup B = \{\omega \in \Omega | \omega \in A \text{ or } \omega \in B\}$
- **Complement**: $A^c = \{\omega \in \Omega | \omega \notin A\}$
- **Difference**: $A \setminus B = \{\omega \in \Omega | \omega \in A \text{ , } \omega \notin B\}$
- **Disjoint Sequence**: $\{A_i\}_{i=1}^n$ if $\forall i \neq j$, $A_i \cap A_j = \emptyset$
- **Relation of Inclusion**: $A \subseteq B$: $\omega \in A \implies \omega \in B$

2.2 Sequence Limits

Definition 3 (Monotone Sequences) *Increasing and decreasing sequences are described as follows:*

1. ***Increasing sequence:*** $A_1 \subseteq A_2 \subseteq \cdots$ *with*

$$\lim_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i$$

2. ***Decreasing sequence:*** $A_1 \supseteq A_2 \supseteq \cdots$ *with*

$$\lim_{n\to\infty} A_n = \bigcap_{i=1}^{\infty} A_i$$

Example 3 *Consider* $A_n = [0, 1/n)$ *for* $n \in \mathbb{N}$*:*

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n = \{0\}$$

This demonstrates how infinite intersections can collapse to single points.

3 Probability

Definition 4 (Probability Axioms) *A probability measure* $P : \mathcal{F} \to [0, 1]$ *satisfies:*

1. ***Non-negativity:*** $P(A) \geq 0 \ \forall A \subseteq \Omega$

2. ***Normalization:*** $P(\Omega) = 1$

3. ***Countable additivity:*** *For disjoint* $\{A_i\}_{i=1}^{n}$*:*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

3.1 Key Properties

Property 1 $P(\emptyset) = 0$

Property 2 $P(A^c) + P(A) = 1$

Property 3 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof 3.1 (Proof of Property 3) *Decompose using disjoint sets:*

$$\begin{aligned}
P(A \cup B) &= P((A \setminus B) \cup (A \cap B) \cup (B \setminus A)) \\
&= P(A \setminus B) + P(A \cap B) + P(B \setminus A) \\
&= P((A \setminus B) \cup (B \setminus A)) + P(B \setminus A) + P(A \cap B) - P(A \cap B) \\
&= P(A) + P(B) - P(A \cup B)
\end{aligned}$$

3.2 Independence

Definition 5 (Independent Events) *Events A and B are independent ($A \perp B$) if:*

$$P(A \cap B) = P(A)P(B)$$

Example 4 (Coin Toss Independence) *For 10 independent tosses, the probability of getting at least one head (T_i indicates that the i-th throw is tail up):*

$$\begin{aligned}
P(\textit{At least 1 H}) &= 1 - P(\textit{All T}) \\
&= 1 - P\left(\bigcap_{i=1}^{10} T_i\right) \\
&= 1 - \prod_{i=1}^{10} P(T_i) \\
&= 1 - \left(\frac{1}{2}\right)^{10} = \frac{1023}{1024}
\end{aligned}$$