Optimization Theory and Algorithm **Lecture 1 - 27/4/2021**

Lecture 1

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1 Part 1: Preliminary

1.1 What is Optimization?

Optimization is a special field that is built on the three intertwined pillars (footsones):

- **Model**: gives rise to optimization problems.
- **Algorithm**: solves optimization problems.
- **Theory**: supports algorithms and models.

We need to remember that

Optimization = Modeling + Algorithm + Theory.

1.2 General Form of Optimization

Definition 1.1. (General Form of Optimization Modeling)

Suppose that $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ is a well-defined function. Then

$$\min_{x} \ f(\mathbf{x}), \tag{1}$$

$$\text{s.t. } \mathbf{x} \in \mathcal{X}, \tag{2}$$

where f is called a s an *objective function*, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top \in \mathcal{X}$ is a *decision variable*, and $\mathcal{X}$ is the so-called *feasible set*. For the feasible set $\mathcal{X}$, it is commonly denoted as

$$\mathcal{X} = \{\mathbf{x} : c_i(\mathbf{x}) \leqslant 0, i = 1, \ldots, l \text{ and } c_j(\mathbf{x}) = 0, j = l+1, \ldots, l+m\},$$

where $c_i(\mathbf{x}) \leqslant 0, i = 1, \ldots, l$ are l *inequality constrains*, and $c_j(\mathbf{x}) = 0, j = l+1, \ldots, l+m$ are m *equality constrains*.

Definition 1.2. (Global Minimum)

Point $\mathbf{x}^* \in \mathcal{X}$ is the global minimum of (1) if for any $\mathbf{x} \in \mathcal{X}$, $f(\mathbf{x}) \geqslant f(\mathbf{x}^*) = f^*$.

Definition 1.3. (Local Minimum)

Point $\mathbf{x}^* \in \mathcal{X}$ is a local minimum of (1) if there exists a neighborhood of $\mathbf{x}^*$, $N(\mathbf{x}^*, \epsilon) = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}^*\| \leqslant \epsilon\}$, such that for any $\mathbf{x} \in N(\mathbf{x}^*, \epsilon)$, $f(\mathbf{x}) \geqslant f(\mathbf{x}^*)$.

For an optimization problem, we may have many local minimum points and global minimum points. Draw an example by yourself!

Q: Give us an optimization example you have learnt with the general optimization formulation in Definition 1.1.

1.3 Modeling in Optimization

Example 1.4. (Transportation Problem in the Operational Management)

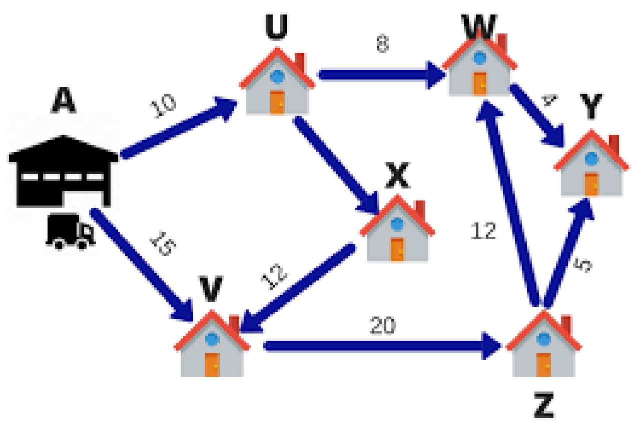

Figure 1: An example of Transportation Problem

Transportation problem (see Figure 1) is a typical problem of operational management where the objective is to minimize the cost of distributing a product form a number of sources or origins to a number of destinations.

Modeling:

- Origin: $O_1, O_2, \ldots, O_m$, and each origin $O_i, i = 1, \ldots, m$ has a supply of a_i units.
- Destination: $D_1, D_2, \ldots, D_n$, and each D_j has a demand for $b_j, j = 1, \ldots, n$ to be delivered from the origins.

- c_{ij} is the cost per unit distributed from the origin O_i to the destination D_j.
- Aim: Finding a set of x_{ij}'s $i = 1, \dots, m; j = 1, \dots, n$ to meet supply and demand requirements at a minimum distribution cost.

Optimization Formulation:

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} c_{ij}, \tag{3}$$

$$\text{s.t. } x_{ij} \geqslant 0, i = 1, \dots, m; j = 1, \dots, n, \tag{4}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \tag{5}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \tag{6}$$

where (4) are the inequality constrains and (5) and (6) are the equality constrains.

Q: Is this the general form of optimization (1.1)?

Q: Why called it as a *Linear Program*?

Example 1.5. (Curve Fitting)

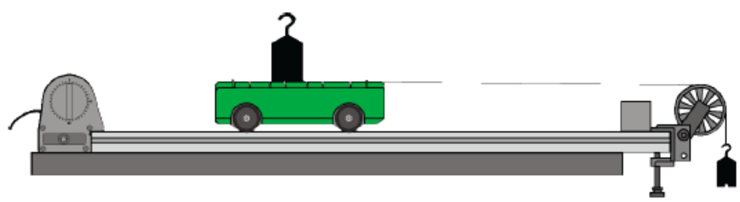

Figure 2: Experiment of Newton's Second Law of Motion

Let us recall the experiment for justifying Newton's Second Law of Motion (see Figure 2).

- Data: $D = \{(a_i, F_i)\}_{i=1}^{n}$, where a_i is the acceleration, and F_i is the corresponding farce.
- Aim: To fit a beset curve for the data we obtained from the experiment of Newton's Second Law of Motion.
- Based on Figure 3, we guess that

  $$F_i = m a_i + \epsilon_i,$$

  where ϵ_i is the noise in the i^{th} experiment and m is the corresponding mass.

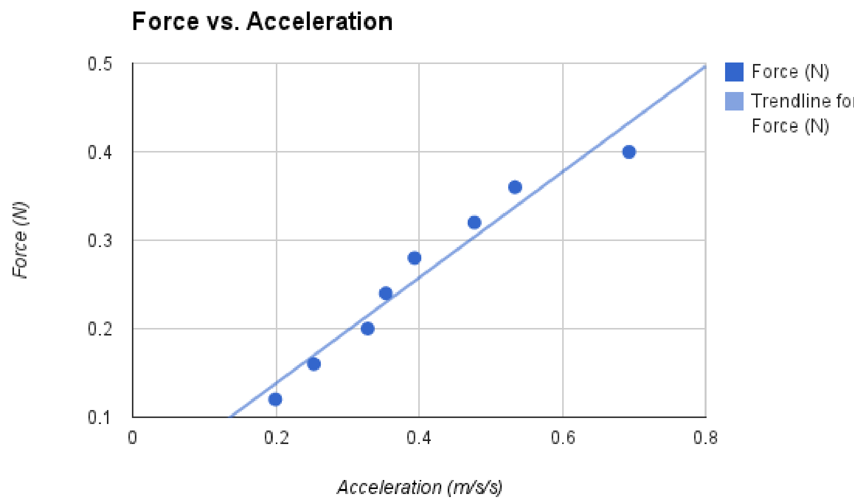

Figure 3: Data of Experiment for Newton's Second Law of Motion

Optimization Formulation:

$$\min_{m} \sum_{i=1}^{n} (F_i - ma_i)^2. \tag{7}$$

This is the so-called **least squares method**.