Optimization Theory and Algorithm **Lecture 8 - 10/13/2021**

Lecture 8

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1 Interior-Point Method for Linear Programming

Recall Simplex method. The fundamental theorem:

Theorem 1.1. *For a standard form LP, if its feasible domain P is nonempty, then the optimal objective value of $z = \mathbf{c}^\top \mathbf{x}$ over P is either unbounded below, or it is attained at (at least) an extreme point of P.*

The simplex method pivots from one point to another better point.

Step1: Find a bfs $\mathbf{x}$ with $\mathbf{A} = [\mathbf{B}|\mathbf{N}]$.

Step2: Check for n.b.v's

$$r_q = \mathbf{c}^T \mathbf{d}_q (= c_q - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_q).$$

If $r_q \geq 0, \ \forall$ nonbasic x_q, then the current bfs is optimal.
Otherwise, pick one $r_q < 0$. Go to next step.

Step3: If $\mathbf{d}_q \geq 0$, then LP is unbounded.
Otherwise, find

$$\alpha = \min_{i:\text{basic}} \{ \frac{x_i}{-d_{q_i}} \mid d_{q_i} < 0 \}.$$

Then $\mathbf{x} \leftarrow \mathbf{x} + \alpha \mathbf{d}_q$ is a new bfs.

Update $\mathbf{B}$ and $\mathbf{N}$. Go to Step 2.

Figure 1: Simplex Method

Revisit the Simplex method by an example:

$$\min -4x_1 - 2x_2 \text{ subject to} \tag{1}$$
$$x_1 + x_2 + x_3 = 5, \tag{2}$$
$$2x_1 + 1/2x_2 + x_4 = 8 \tag{3}$$
$$x \geqslant 0 \tag{4}$$

In this case, we have

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1/2 & 0 & 1 \end{bmatrix} \tag{5}$$

The basis $\mathcal{B} = \{3, 4\}$, for which we have

$$x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 1 \\ 2 & 1/2 \end{bmatrix}. \tag{6}$$

It holds that

$$-B^{-1}N = \begin{bmatrix} -1 & -1 \\ -2 & -1/2 \end{bmatrix} \quad d_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \begin{matrix} x_3 \\ x_4 \\ x_1 \\ x_2 \end{matrix} \quad c^\top d_1 = -4 < 0, \quad \lambda = \min\left(\frac{8}{2}, \frac{5}{1}\right) = 4. \tag{7}$$

$$x_B = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{matrix} x_1 \\ x_3 \end{matrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \quad -B^{-1}N = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & -\frac{1}{2} \end{bmatrix} \tag{8}$$

$$d_2 = \begin{bmatrix} -\frac{1}{4} \\ -\frac{3}{4} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} \quad c^\top d_2 = 1 - 2 < 0, \quad d_4 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{bmatrix} \quad c^\top d_4 = 2 > 0 \tag{9}$$

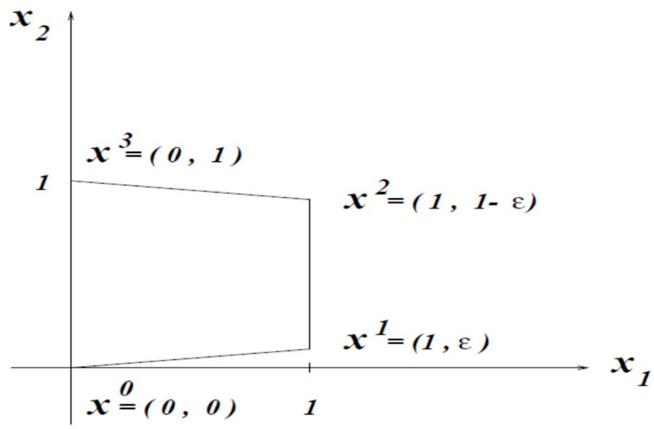
$$x_B = \begin{bmatrix} \frac{11}{3} \\ \frac{4}{3} \end{bmatrix} \begin{matrix} x_1 \\ x_2 \end{matrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 1/2 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B^{-1}N = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{4}{3} & -\frac{2}{3} \end{bmatrix} \tag{10}$$

$$c^\top d_3 = 4/3 > 0 \quad c^\top d_4 = 4/3 > 0. \tag{11}$$

Done!

Problem: Klee and Minty [1972] an n-dimensional problem, 2^n vertices, visit all points! That is simplex method is not a polynomial algorithm. Simplex method is proposed in 1948. It is not easy to find a problem.

$$
\begin{aligned}
(2 \text{ dim}) \quad \min \quad & -x_2 \\
\text{s. t.} \quad & x_1 \geq 0 \\
& x_1 \leq 1 \\
& x_2 \geq \epsilon x_1 \qquad (0 < \epsilon < \tfrac{1}{2}) \\
& x_2 \leq 1 - \epsilon x_1 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



(b) $x^0 \to x^1 \to x^2 \to x^3$

Figure 2: Counter Example for 2D

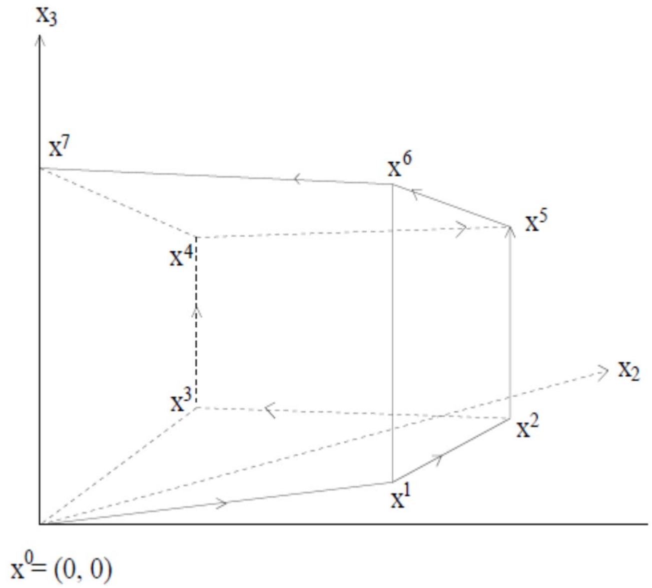
The Lagrangian is

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \mathbf{c}^\top \mathbf{x} - \lambda^\top (A\mathbf{x} - \mathbf{b}) - s^\top \mathbf{x}.$$

The KKT conditions of the standard linear programming are

$$
\begin{aligned}
A^\top \lambda + s &= \boldsymbol{\lambda}, \\
A\mathbf{x} &= \mathbf{b}, \\
\mathbf{x} &\succeq 0, \\
x_i s_i &= 0, \\
s_i &\succeq 0.
\end{aligned}
$$

$$
\begin{aligned}
(3\ \text{dim})\quad \min\quad & -x_3 \\
\text{s. t.}\quad & x_1 \geq 0 \\
& x_1 \leq 1 \\
& x_2 \geq \epsilon x_1 \\
& x_2 \leq 1 - \epsilon x_1 \\
& x_3 \geq \epsilon x_2 \\
& x_3 \leq 1 - \epsilon x_2 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$



(b) $x^0 \to x^1 \to x^2 \to x^3 \cdots \to x^7$

Figure 3: Counter Example for 3D

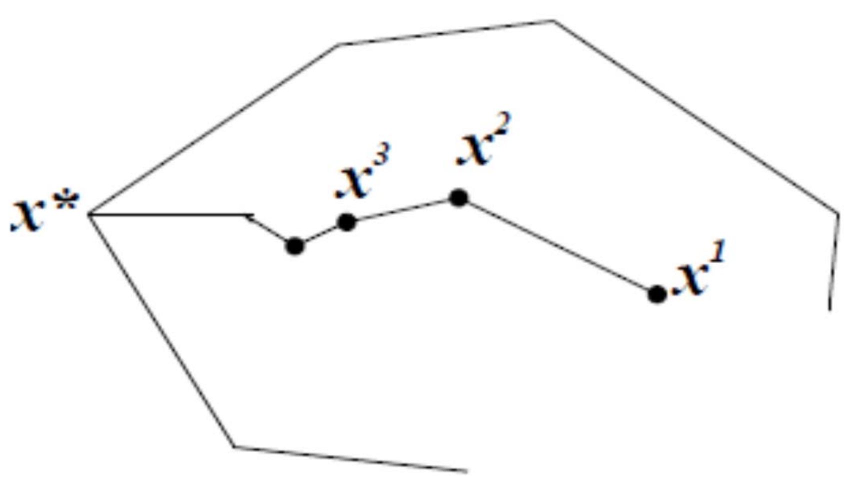


Figure 4: Path of Interior Point Method

Given (x^0, λ^0, s^0) with $(x^0, s^0) > 0$;
for $k = 0, 1, 2, \dots$

Choose $\sigma_k \in [0, 1]$ and solve

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ S^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta s^k \end{bmatrix} = \begin{bmatrix} -r_c^k \\ -r_b^k \\ -X^k S^k e + \sigma_k \mu_k e \end{bmatrix},$$

where $\mu_k = (x^k)^T s^k / n$;

Set

$$(x^{k+1}, \lambda^{k+1}, s^{k+1}) = (x^k, \lambda^k, s^k) + \alpha_k(\Delta x^k, \Delta \lambda^k, \Delta s^k),$$

choosing α_k so that $(x^{k+1}, s^{k+1}) > 0$.

end (for).

Figure 5: Interior Point Method